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Existence and uniqueness of positive solutions to three-point boundary value problems for second order impulsive differential equations

Chen Yang[a]*, Jurang Yan[b]

a. Department of Mathematics, Business College of Shanxi University, Taiyuan 030031, P.R. China

b. School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P.R. China

Abstract Using a fixed point theorem of generalized concave operators, we present in this paper criteria which guarantee the existence and uniqueness of positive solutions to three-point boundary value problems for second order impulsive differential equations.

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1 Introduction

In this paper, we study the existence and uniqueness of positive solutions to the following three-point boundary value problems for second order impulsive differential equations:

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \ t \neq t_k, k = 1, 2, \cdots, m, \\ \Delta x'|_{t=t_k} = I_k(x(t_k)), k = 1, 2, \cdots, m, \\ x'(0) = 0, \beta x(\eta) = x(1), \end{cases} \tag{1.1}$$

where $f \in C[J \times \mathbf{R}, \mathbf{R}], J = [0, 1], 0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < 1$, $\Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-), x'(t_k^+), x'(t_k^-)$ denote the right limit(left limit) of $x'(t)$ at $t = t_k$ respectively. $I_k \in C[\mathbf{R}, \mathbf{R}], k = 1, 2, \cdots, m$ and $\eta \in (0, 1)$, $0 < \beta < 1$.

Impulsive differential equations have been studied extensively in recent years. The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equations in $\mathbf{R}^n$, see

*This research is supported by Science Foundation of Business College of Shanxi University(2008053). Corresponding author. E-mail address: yangchen0809@126.com (C.Yang).

[3,17,26] and the references therein. The theory of impulsive differential equations has become an important area of investigation in recent years and is much richer than the corresponding theory of differential equations (see for instance [1,4-10,12-15,18-21,23-25] and their references). Second-order impulsive differential equations have been studied by many authors with much of the attention given to positive solutions. For a small sample of such work, we refer the reader to works by Agarwal, O'Regan [1], Feng, Xie [9], Hu et al. [12], Jankowski [14,15], Lee [18], Lin, Jiang [19], Liu et al. [20], Wang et al. [27] and Zhang [30]. The results of these papers are based on Schauder fixed point theorem, Leggett-Williams theorem, fixed point index theorems in cones, Krasnoselskii's fixed point theorem, the method of upper-lower solutions, fixed point theorems in cones and so on. However, few papers can be found in the literature on the existence of positive solutions to three-point boundary value problems for second-order impulsive differential equations. Three-point boundary value problems for differential equations or difference equations have been studied by many authors with much of the attention given to positive solutions. Here we mention only a few of them, see for example papers by Ahmad, Nieto [2], Gupta and Trofimchuk [11], Karaca [16], Ma [22] and Yang, Zhai and Yan [28].

To the best of our knowledge, no paper can be found in the literature on the existence and uniqueness of positive solutions to three-point boundary value problems for second-order impulsive differential equations. In this paper, we shall study the problem (1.1) and not suppose the existence of upper-lower solutions. Different from the above works mentioned, in this paper we will use a fixed point theorem of generalized concave operators to show the existence and uniqueness of positive solutions for the problem (1.1).

For convenience, we list the following assumptions on the functions $f(t,x), I_k(x)$:

(H_1) $f(t,0)\geq 0, f\left(t,\frac{\beta(1-\eta^2)}{2(1-\beta)}\right)>0, t\in[0,1]$and $f(t,x)$ is increasing in $x\in[0,\infty)$ for each $t\in[0,1]$;

(H_2) $I_k(0)\leq 0$ and $I_k(x)$ is decreasing in $x\in[0,\infty), k=1,2,\cdots,m$;

(H_3) for any $\lambda\in(0,1)$ and $x\geq 0$, there exist $\alpha_1(\lambda),\alpha_2(\lambda)\in(\lambda,1)$ such that

$$f(t,\lambda x)\geq\alpha_1(\lambda)f(t,x), I_k(\lambda x)\leq\alpha_2(\lambda)I_k(x), k=1,2,\cdots,m.$$

(H_4)

$$\sum_{k=1}^{m}(1-t_k)I_k\left(\frac{1-\beta\eta^2}{2(1-\beta)}\right)<0.$$

2 Preliminaries

In this section, we state some definitions, notations and known results. For convenience of readers, we suggest that one refer to [29] and references therein for details.

Suppose that $(E,\|\cdot\|)$ is a real Banach space which is partially ordered by a cone $P\subset E, i.e., x\leq y$ if and only if $y-x\in P$. By θ we denote the zero element of E. Recall that a non-empty closed convex set $P\subset E$ is called a cone if it satisfies (i) $x\in P,\lambda\geq 0\Rightarrow\lambda x\in P$; (ii) $x\in P,-x\in P\Rightarrow x=\theta$.

Moreover, P is called normal if there exists a constant $N>0$ such that, for all $x,y\in E$, $\theta\leq$

$x \leq y$ implies $\|x\| \leq N\|y\|$; in this case N is called the normality constant of P. We say that an operator $A : E \to E$ is increasing(decreasing) if $x \leq y$ implies $Ax \leq Ay(Ax \geq Ay)$.

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta(i.e., h \geq \theta$ and $h \neq \theta)$, we denote by P_h the set $P_h = \{x \in E|\ x \sim h\}$. Clearly, $P_h \subset P$ is convex and $aP_h = P_h$ for all $a > 0$.

We now present a fixed point theorem of generalized concave operators which will be used in the latter proof. See [29] for further information.

Theorem 2.1(from the Lemma 2.1 and Theorem 2.1 in [29]). Let $h > \theta$ and P be a normal cone. Assume that: (D_1) $A : P \to P$ is increasing and $Ah \in P_h$; (D_2) For any $x \in P$ and $t \in (0, 1)$, there exists $\alpha(t) \in (t, 1)$ such that $A(tx) \geq \alpha(t)Ax$. Then (i) there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 < v_0,\ u_0 \leq Au_0 \leq Av_0 \leq v_0$; (ii) operator equation $x = Ax$ has a unique solution in P_h.

Remark 2.2. An operator A is said to be **generalized concave** if A satisfies condition (D_2)(see Remark 1.1 in [29]).

In what follows, for the sake of convenience, let $J' = J\backslash\{t_1, t_2, \cdots, t_m\}$, $C[J, \mathbf{R}] = \{x|x : J \to \mathbf{R}$ is continuous $\}$, $PC^1[J, \mathbf{R}] = \{x \in C[J, \mathbf{R}]|x'(t)$ is continuous at $t \neq t_k$ and left continuous at $t = t_k,\ x'(t_k^+)$ exists, $k = 1, 2, \cdots, m\}$. Evidently, $C[J, \mathbf{R}]$ is a Banach space with the norm $\|x\|_C = \sup\{|x(t)| : t \in J\}$ and $PC^1[J, \mathbf{R}]$ is a Banach space with the norm $\|x\|_{PC^1} = \sup\{\|x\|_C, \|x'\|_C\}$.

Definition 2.3. A function $x \in PC^1[J, \mathbf{R}] \bigcap C^2[J', \mathbf{R}]$ is called a solution of the problem (1.1), if it satisfies the problem (1.1).

Lemma 2.4. $x \in PC^1[J, \mathbf{R}] \bigcap C^2[J', \mathbf{R}]$ is a solution of the problem (1.1) if and only if $x \in PC^1[J, \mathbf{R}]$ is the solution of the following integral equation:

$$\begin{aligned} x(t) &= \frac{1}{1-\beta}\int_0^1 (1-s)f(s, x(s))ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta - s)f(s, x(s))ds \\ &- \int_0^t (t-s)f(s, x(s))ds + \sum_{0<t_k<t}(t-t_k)I_k(x(t_k)) \\ &+ \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta - t_k)I_k(x(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m (1-t_k)I_k(x(t_k)). \end{aligned} \tag{2.1}$$

Proof. First suppose that $x \in PC^1[J, \mathbf{R}] \bigcap C^2[J', \mathbf{R}]$ is a solution of the problem (1.1). It is easy to see by integration of (1.1) that

$$\begin{aligned} x'(t) &= x'(0) - \int_0^t f(s, x(s))ds + \sum_{0<t_k<t}[x'(t_k^+) - x'(t_k)] \\ &= -\int_0^t f(s, x(s))ds + \sum_{0<t_k<t} I_k(x(t_k)). \end{aligned}$$

Integrate again, we can get

$$x(t) = x(0) - \int_0^t (t-s) f(s, x(s)) ds + \sum_{0<t_k<t} I_k(x(t_k))(t - t_k). \tag{2.2}$$

Letting $t = 1$ and $t = \eta$ in (2.2), we find

$$x(1) = x(0) - \int_0^1 (1-s) f(s, x(s)) ds + \sum_{k=1}^{m} I_k(x(t_k))(1 - t_k),$$

$$x(\eta) = x(0) - \int_0^\eta (\eta - s) f(s, x(s)) ds + \sum_{0<t_k<\eta} I_k(x(t_k))(\eta - t_k).$$

From the boundary condition $x(1) = \beta x(\eta)$, we have

$$\begin{aligned} x(0) &= \frac{1}{1-\beta} \int_0^1 (1-s) f(s, x(s)) ds - \frac{\beta}{1-\beta} \int_0^\eta (\eta - s) f(s, x(s)) ds \\ &- \frac{1}{1-\beta} \sum_{k=1}^{m} I_k(x(t_k))(1 - t_k) + \frac{\beta}{1-\beta} \sum_{0<t_k<\eta} I_k(x(t_k))(\eta - t_k). \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we have

$$\begin{aligned} x(t) &= \frac{1}{1-\beta} \int_0^1 (1-s) f(s, x(s)) ds - \frac{\beta}{1-\beta} \int_0^\eta (\eta - s) f(s, x(s)) ds \\ &- \int_0^t (t-s) f(s, x(s)) ds + \sum_{0<t_k<t} (t - t_k) I_k(x(t_k)) \\ &+ \frac{\beta}{1-\beta} \sum_{0<t_k<\eta} (\eta - t_k) I_k(x(t_k)) - \frac{1}{1-\beta} \sum_{k=1}^{m} (1 - t_k) I_k(x(t_k)). \end{aligned}$$

Thus, the proof of sufficient is complete.

Conversely, if x is a solution of (2.1). Direct differentiation of (2.1) implies, for $t \neq t_k$

$$x'(t) = -\int_0^t f(s, x(s)) ds + \sum_{0<t_k<t} I_k(x(t_k)).$$

Further

$$x''(t) = -f(t, x(t)), \ \Delta x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-) = I_k(x(t_k)).$$

So $x \in C^2[J', \mathbf{R}]$ and it is easy to verify that $x'(0) = 0, x(1) = \beta x(\eta)$, and the lemma is proved.□

Define an operator $A : C[J, \mathbf{R}] \to C[J, \mathbf{R}]$ by

$$\begin{aligned} Ax(t) &= \frac{1}{1-\beta} \int_0^1 (1-s) f(s, x(s)) ds - \frac{\beta}{1-\beta} \int_0^\eta (\eta - s) f(s, x(s)) ds \\ &- \int_0^t (t-s) f(s, x(s)) ds + \sum_{0<t_k<t} (t - t_k) I_k(x(t_k)) \\ &+ \frac{\beta}{1-\beta} \sum_{0<t_k<\eta} (\eta - t_k) I_k(x(t_k)) - \frac{1}{1-\beta} \sum_{k=1}^{m} (1 - t_k) I_k(x(t_k)). \end{aligned}$$

Lemma 2.5. $x \in PC^1[J, \mathbf{R}] \bigcap C^2[J', \mathbf{R}]$ is a solution of the problem (1.1) if and only if $x \in PC^1[J, \mathbf{R}]$ is a fixed point of the operator A.

3 Existence and uniqueness of positive solutions for problem (1.1)

In this section, we apply Theorem 2.1 to study the problem (1.1) and we obtain a new result on the existence and uniqueness of positive solutions. The method used in this paper is new to the literature and so is the existence and uniqueness result to the second-order impulsive differential equations. This is also the main motivation for the study of (1.1) in the present work.

Set $\tilde{P} = \{x \in C[J, \mathbf{R}] | x(t) \geq 0,\ t \in J\}$, the standard cone. It is clear that $\tilde{P}$ is a normal cone in $C[J, \mathbf{R}]$ and the normality constant is 1. Our main result is summarized in the following theorem.

Theorem 3.1. Assume that $(H_1) - (H_4)$ hold. Then (i) there exist $u_0, v_0 \in \tilde{P}_h$ such that

$$\begin{aligned} u_0(t) &\leq \frac{1}{1-\beta}\int_0^1 (1-s)f(s,u_0(s))ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)f(s,u_0(s))ds \\ &- \int_0^t (t-s)f(s,u_0(s))ds + \sum_{0<t_k<t}(t-t_k)I_k(u_0(t_k)) \\ &+ \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(u_0(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m (1-t_k)I_k(u_0(t_k)), t \in J, \end{aligned}$$

$$\begin{aligned} v_0(t) &\geq \frac{1}{1-\beta}\int_0^1 (1-s)f(s,v_0(s))ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)f(s,v_0(s))ds \\ &- \int_0^t (t-s)f(s,v_0(s))ds + \sum_{0<t_k<t}(t-t_k)I_k(v_0(t_k)) \\ &+ \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(v_0(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m (1-t_k)I_k(v_0(t_k)), t \in J; \end{aligned}$$

(ii) the problem (1.1) has a unique positive solution x^* in $\tilde{P}_h \bigcap PC^1[J, \mathbf{R}]$, where

$$h(t) = -\frac{1}{2}t^2 + \frac{1-\beta\eta^2}{2(1-\beta)}, t \in [0,1].$$

Remark 3.1. It is easy to see that the function $h(t)$ satisfies $h'(0) = 0,\ \beta h(\eta) = h(1),\ h''(t) \equiv -1$ and for $t \in [0,1]$

$$h(t) = \frac{1}{1-\beta}\int_0^1 (1-s)ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)ds - \int_0^t (t-s)ds,$$

$$0 < \frac{\beta(1-\eta^2)}{2(1-\beta)} = h(1) \leq h(t) \leq h(0) = \frac{1-\beta\eta^2}{2(1-\beta)}.$$

Proof of Theorem 3.1 Firstly, we show that $A : \tilde{P} \to \tilde{P}$ is increasing, generalized concave. To illuminate this, we divide into two cases: (i) for any $t \in [0, \eta]$, we have

$$\begin{aligned} Ax(t) &= \frac{1}{1-\beta}\left[\int_0^t (1-s)f(s,x(s))ds + \int_t^\eta (1-s)f(s,x(s))ds + \int_\eta^1 (1-s)f(s,x(s))ds\right] \\ &- \frac{\beta}{1-\beta}\left[\int_0^t (\eta-t)f(s,x(s))ds + \int_t^\eta (\eta-s)f(s,x(s))ds\right] - \int_0^t (t-s)f(s,x(s))ds \end{aligned}$$

$$\begin{aligned}
&+ \sum_{0<t_k<t}(t-t_k)I_k(x(t_k)) + \frac{\beta}{1-\beta}\left[\sum_{0<t_k<t}(\eta-t_k)I_k(x(t_k)) + \sum_{t\le t_k<\eta}(\eta-t_k)I_k(x(t_k))\right]\\
&- \frac{1}{1-\beta}\left[\sum_{0<t_k<t}(1-t_k)I_k(x(t_k)) + \sum_{t\le t_k<\eta}(1-t_k)I_k(x(t_k)) + \sum_{\eta\le t_k<1}(1-t_k)I_k(x(t_k))\right]\\
&= \frac{1}{1-\beta}\left[\int_0^t(1-t-\beta\eta+\beta t)f(s,x(s))ds + \int_t^\eta(1-s-\beta\eta+\beta s)f(s,x(s))ds\right.\\
&+ \left.\int_\eta^1(1-s)f(s,x(s))ds\right] - \frac{1}{1-\beta}\left[\sum_{0<t_k<t}(1-t-\beta\eta+\beta t)I_k(x(t_k))\right.\\
&+ \left.\sum_{t\le t_k<\eta}(1-t_k-\beta\eta+\beta t_k))I_k(x(t_k)) + \sum_{\eta\le t_k<1}(1-t_k)I_k(x(t_k))\right].
\end{aligned}$$

(ii) for any $t\in(\eta,1]$, we have

$$\begin{aligned}
Ax(t) &= \frac{1}{1-\beta}\left[\int_0^\eta(1-s)f(s,x(s))ds + \int_\eta^t(1-s)f(s,x(s))ds + \int_t^1(1-s)f(s,x(s))ds\right]\\
&- \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)f(s,x(s))ds - \left[\int_0^\eta(t-s)f(s,x(s))ds + \int_\eta^t(t-s)f(s,x(s))ds\right]\\
&+ \left[\sum_{0<t_k<\eta}(t-t_k)I_k(x(t_k)) + \sum_{\eta\le t_k<t}(t-t_k)I_k(x(t_k))\right] + \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(x(t_k))\\
&- \frac{1}{1-\beta}\left[\sum_{0<t_k<\eta}(1-t_k)I_k(x(t_k)) + \sum_{\eta\le t_k<t}(1-t_k)I_k(x(t_k)) + \sum_{t\le t_k<1}(1-t_k)I_k(x(t_k))\right]\\
&= \frac{1}{1-\beta}\left[\int_0^\eta(1-t-\beta\eta+\beta t)f(s,x(s))ds + \int_\eta^t(1-t-\beta s+\beta t)f(s,x(s))ds\right.\\
&+ \left.\int_t^1(1-s)f(s,x(s))ds\right] - \frac{1}{1-\beta}\left[\sum_{0<t_k<\eta}(1-t-\beta\eta+\beta t)I_k(x(t_k))\right.\\
&+ \left.\sum_{\eta\le t_k<t}(1-t-\beta t_k+\beta t)I_k(x(t_k)) + \sum_{t\le t_k<1}(1-t_k)I_k(x(t_k))\right].
\end{aligned}$$

For case (i), we can easily get $1-t-\beta\eta+\beta t\ge 0$ for $t\in[0,\eta]$, $1-s-\beta\eta+\beta s\ge 0$ for $s\in[t,\eta]$ and $1-t_k-\beta\eta+\beta t_k\ge 0$ for $t_k\in[t,\eta)$. For case (ii), we can easily get $1-t-\beta\eta+\beta t\ge 0$ for $t\in(\eta,1]$, $1-t-\beta s+\beta t\ge 0$ for $s\in[\eta,t]$ and $1-t-\beta t_k+\beta t\ge 0$ for $t_k\in[\eta,t)$. Note that $\eta\in(0,1)$, $0<\beta<1$, and from $(H_1),(H_2)$, we obtain

$$\frac{1}{1-\beta}\int_0^1(1-s)f(s,x(s))ds - \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)f(s,x(s))ds - \int_0^t(t-s)f(s,x(s))ds \ge 0, \quad (3.1)$$

$$\sum_{0<t_k<t}(t-t_k)I_k(x(t_k)) + \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(x(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m(1-t_k)I_k(x(t_k)) \ge 0. \quad (3.2)$$

So we have $Ax(t)\ge 0,\ t\in[0,1]$. Further, also from the above two cases (i),(ii) and $(H_1),(H_2)$, we can easily prove that $A:\tilde P\to\tilde P$ is increasing. Set $\alpha(t)=\min\{\alpha_1(t),\alpha_2(t)\}, t\in(0,1)$. Then

$\alpha(t) \in (t,1)$. For any $\lambda \in (0,1)$ and $x \in \tilde{P}$, it follows from the above two cases (i),(ii), (3.1),(3.2) and (H_3) that

$$\begin{aligned} A(\lambda x)(t) &\geq \alpha_1(\lambda)\left[\frac{1}{1-\beta}\int_0^1(1-s)f(s,x(s))ds - \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)f(s,x(s))ds\right. \\ &- \left.\int_0^t(t-s)f(s,x(s))ds\right] + \alpha_2(\lambda)\left[\sum_{0<t_k<t}(t-t_k)I_k(x(t_k)) + \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(x(t_k))\right. \\ &- \left.\frac{1}{1-\beta}\sum_{k=1}^m(1-t_k)I_k(x(t_k))\right] \geq \alpha(\lambda)\left\{\frac{1}{1-\beta}\int_0^1(1-s)f(s,x(s))ds\right. \\ &- \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)f(s,x(s))ds - \int_0^t(t-s)f(s,x(s))ds + \sum_{0<t_k<t}(t-t_k)I_k(x(t_k)) \\ &+ \left.\frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(x(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m(1-t_k)I_k(x(t_k))\right\} = \alpha(\lambda)Ax(t). \end{aligned}$$

That is, $A(\lambda x) \geq \alpha(\lambda)Ax, x \in \tilde{P}, \lambda \in (0,1)$. So $A : \tilde{P} \to \tilde{P}$ is generalized concave.

Secondly, we prove $Ah \in \tilde{P}_h$. Set

$$r_1 = \min_{t\in[0,1]} f\left(t, \frac{\beta(1-\eta^2)}{2(1-\beta)}\right),\ r_2 = \max_{t\in[0,1]} f\left(t, \frac{1-\beta\eta^2}{2(1-\beta)}\right).$$

Then from (H_1), we have $r_2 \geq r_1 > 0$. Further, from $(H_1),(H_2)$, the above two cases (i),(ii) and (3.2), we have

$$\begin{aligned} Ah(t) &\geq \frac{1}{1-\beta}\int_0^1(1-s)f(s,h(1))ds - \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)f(s,h(1))ds - \int_0^t(t-s)f(s,h(1))ds \\ &\geq r_1\left[\frac{1}{1-\beta}\int_0^1(1-s)ds - \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)ds - \int_0^t(t-s)ds\right] = r_1h(t), \end{aligned}$$

From (H_2), we have

$$\sum_{0<t_k<t}(t-t_k)I_k(h(t_k)) + \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(h(t_k)) \leq 0,\ -\frac{1}{1-\beta}\sum_{k=1}^m(1-t_k)I_k(h(t_k)) \geq 0. \quad (3.3)$$

Further, from $(H_1),(H_2),(H_4)$, the above two cases (i),(ii) and (3.1)-(3.3), we have

$$\begin{aligned} Ah(t) &\leq \frac{1}{1-\beta}\int_0^1(1-s)f(s,h(0))ds - \frac{\beta}{1-\beta}\int_0^\eta(\eta-s)f(s,h(0))ds \\ &- \int_0^t(t-s)f(s,h(0))ds - \frac{1}{1-\beta}\sum_{k=1}^m(1-t_k)I_k(h(0)) \\ &\leq r_2h(t) - \frac{1}{1-\beta}\sum_{k=1}^m(1-t_k)I_k\left(\frac{1-\beta\eta^2}{2(1-\beta)}\right) \\ &\leq \left[r_2 - \frac{2}{\beta(1-\eta^2)}\sum_{k=1}^m(1-t_k)I_k\left(\frac{1-\beta\eta^2}{2(1-\beta)}\right)\right]h(t). \end{aligned}$$

Hence,

$$r_1h \leq Ah \leq \left[r_2 - \frac{2}{\beta(1-\eta^2)}\sum_{k=1}^m(1-t_k)I_k\left(\frac{1-\beta\eta^2}{2(1-\beta)}\right)\right]h.$$

That is, $Ah \in \tilde{P}_h$. Finally, an application of Theorem 2.1 implies that (i) there are $u_0, v_0 \in \tilde{P}_h$ such that $u_0 \leq Au_0, Av_0 \leq v_0$; (ii) operator equation $x = Ax$ has a unique solution in $\tilde{P}_h$. That is,

$$\begin{aligned}
u_0(t) \;\leq\; & \frac{1}{1-\beta}\int_0^1 (1-s)f(s,u_0(s))ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)f(s,u_0(s))ds \\
- & \int_0^t (t-s)f(s,u_0(s))ds + \sum_{0<t_k<t}(t-t_k)I_k(u_0(t_k)) \\
+ & \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(u_0(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m (1-t_k)I_k(u_0(t_k)), t \in J,
\end{aligned}$$

$$\begin{aligned}
v_0(t) \;\geq\; & \frac{1}{1-\beta}\int_0^1 (1-s)f(s,v_0(s))ds - \frac{\beta}{1-\beta}\int_0^\eta (\eta-s)f(s,v_0(s))ds \\
- & \int_0^t (t-s)f(s,v_0(s))ds + \sum_{0<t_k<t}(t-t_k)I_k(v_0(t_k)) \\
+ & \frac{\beta}{1-\beta}\sum_{0<t_k<\eta}(\eta-t_k)I_k(v_0(t_k)) - \frac{1}{1-\beta}\sum_{k=1}^m (1-t_k)I_k(v_0(t_k)), t \in J;
\end{aligned}$$

and the problem (1.1) has a unique solution x^* in $\tilde{P}_h$. Moreover, from Lemmas 2.4 and 2.5 we know that $x^* \in PC^1[J, \mathbf{R}]$. Evidently, x^* is a positive solution of the problem (1.1).$\Box$

Remark 3.2. For the case of $I_k = 0, k = 1, 2, \ldots, m$, the problem (1.1) reduces to the following three-point boundary value problem for ordinary differential equations:

$$\begin{cases} x''(t) + f(t, x(t)) = 0, \ 0 < t < 1, \\ x'(0) = 0, \beta x(\eta) = x(1). \end{cases} \tag{3.4}$$

We can establish the existence and uniqueness of positive solutions for the problem (3.1) by the same method used in this paper, which is new to the literature. So the method employed in this paper is different from previous ones in literature and the result obtained in this paper is new.

4 An example

To illustrate how our main result can be used in practice we present an example.

Example 4.1. Consider the following boundary value problem

$$\begin{cases} x''(t) + x^\gamma + \psi(t) = 0, \ t \in J, t \neq \frac{1}{2}, \\ \Delta x'|_{t=\frac{1}{2}} = -\sqrt[4]{x(\frac{1}{2})}, \\ x'(0) = 0, \frac{1}{2}x(\frac{1}{4}) = x(1), \end{cases} \tag{4.1}$$

where $\gamma \in (0, 1)$ and $\psi : [0, 1] \to [0, +\infty)$ is a continuous function.

Conclusion. The impulsive problem (4.1) has a unique positive solution in $\tilde{P}_h \bigcap PC^1[J, \mathbf{R}]$, where

$$h(t) = -\frac{1}{2}t^2 + \frac{1-\beta\eta^2}{2(1-\beta)} = -\frac{1}{2}t^2 + \frac{31}{32}, t \in [0, 1].$$

Proof. The problem (4.1) can be regarded as a boundary value problem of the form (1.1), where $\eta = \frac{1}{4},\ \beta = \frac{1}{2},\ t_1 = \frac{1}{2}, f(t,x) = x^\gamma + \psi(t),\ I_1(x) = -x^{\frac{1}{4}}$. It is not difficult to see that the conditions $(H_1), (H_2)$ and (H_4) hold. In addition, let $\alpha_1(\lambda) = \lambda^\gamma, \alpha_2(\lambda) = \lambda^{\frac{1}{4}}$. Then, the condition (H_3) of Theorem 3.1 holds. Hence, by Theorem 3.1, the conclusion follows, and the proof is complete.

Remark 4.1. Example 4.1 implies that there is a large number of functions that satisfy the conditions of Theorem 3.1. In addition, the conditions of Theorem 3.1 are also easy to check.

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